carborundum when used with a "whisker" may *either* decrease *or* increase with the increase of temperature of the crystal. We have observed these features, and it seems likely that Sen (Ind. Jour. Physics, vol. x. 1936) has also noticed them.

Physics Laboratory, Dacca University.
Nov. 15th, 1936.

XIII. *The Diffraction of Electromagnetic Waves from an Electrical Point Source round a Finitely Conducting Sphere, with Applications to Radiotelegraphy and the Theory of the Rainbow.*—Part I. *By* Balth. van der Pol *and* H. Bremmer, *Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven, Holland*.*

1. *Introduction.*

THE paper concerns the calculation of the electromagnetic field due to an oscillating electrical dipole placed outside a finitely conducting sphere. This is one of the fundamental problems of radiotelegraphy and has aroused new interest by the coming of television. Here the radiating dipole represents the transmitter antenna, and the absorbing sphere the earth. In our calculations the atmosphere is taken homogeneous and any effects due to the ionosphere are therefore neglected. (The chief contents of Part I. of this investigation have been communicated by Balth. van der Pol in a lecture before the Institute of Radio Engineers in New York in November 1936.)

The rigid solution for a given wave-length, based on the Maxwellian equations with proper boundary conditions, leads to an infinite series of spherical harmonics with coefficients containing twelve Bessel functions, which has been known already for a long time and which is given below as formula (1).

As in all diffraction problems the character of the solution is determined by the ratio of the dimensions of the obstacle to the wave-length λ, in case $2\pi a/\lambda$, where a is the radius of the sphere. When this ratio is small, as, for instance, in the problem of the scattering of light by submicroscopic particles, the series (1) converges so rapidly

* Communicated by Dr. Balth. van der Pol.

that a few terms only give a sufficient numerical approximation. For instance, this approximation led Lord Rayleigh [1] to discover his λ^{-4} law for scattered light, and it also is at the base of Debye's investigation of the light pressure [2].

In the radio application of our problem this ratio is of the order 10^3 to 10^8, while the corresponding ratio in the theory of the rainbow is determined by the dimensions of the raindrops and the wave-length of light, leading to a value of $2\pi a/\lambda$ of the order of 10^4. Under these circumstances the series (1) converges in general extremely slowly, the main contribution being given by the terms where n is of the order $2\pi a/\lambda$. Lord Rayleigh [3] evaluated for an analogous acoustical problem with a similar, but slightly different boundary condition, the field at the surface of the sphere, the point source being also on the surface, by simply adding numerically the important terms for the cases $2\pi a/\lambda=2$ and 10. Later on, A. E. H. Love [4] even undertook the laborious task of evaluating the series (1) for the same case of radiotelegraphy with $2\pi a/\lambda=8000$, by numerical addition of the terms of the series taken in groups. In his paper an historical summary of the problem up to 1915 is also given.

Considerable progress in the present problem was subsequently obtained by G. N. Watson [5], who transformed (1), with the aid of a contour integration, into another much more rapidly converging series (15), which was further considered in more detail with a view to the application to radio by one of us [6]. Also Laporte's [7] investigation uses this series, and a clear exposition by A. Sommerfeld of Watson's method is to be found in Frank-Von Mises, *Differential Gleichungen der Physik*, II, 2. vermehrte Auflage, pp. 964–977. Finally, T. L. Eckersley [8] treated the problem with his method of phase integrals, while P. S. Epstein [9] simplified the treatment by an application of Huygens's principle.

The present investigation attacks the problem in two quite different ways :—

I. The first one, extending Watson's treatment, is more appropriate for investigating the disturbance beyond the optical horizon of the point source and is pushed to considerable numerical details while considering a.o. in the radio case the effect of the height of the trans-

mitter above the earth. *E. g.*, we find that for practical cases of modern television transmission ($\lambda=7$ meter) the attenuation of the radiowaves due to the finite conductivity of the earth dominates *by far* the attenuation due to diffraction. Further, when moving along the surface of the earth, the optical horizon is found not to be a very critical point as regards the amplitude of waves of the order of $\lambda=7$ meter, more or less sharp shadows only occurring for much shorter wave-lengths. Also it is found that for this wave-length the factor by which the field on the surface of the earth and beyond the horizon is increased due to raising the transmitter to a certain height is, as a first approximation, independent of the distance. Moreover, the numerical results show that, as in the analogous problem for a plane boundary, the conductivity of the earth has a very pronounced influence on the attenuation of the waves.

II. The second method also starts with the rigorous harmonic series (1), being a solution of the wave equation. It appeared possible to split up this series uniquely in a series of an infinite number of other series, each of which may be considered to represent the rigorous expression of what can be visualized as rays undergoing successive reflexions at the inner surface of the sphere. An approximation of each of these series clearly shows their ray-character in complete accordance with the classical Airy ray-treatment of the theory of the rainbow. The present investigation therefore provides the link between the exact wave solution and the approximate ray solution of the rainbow extending the latter to the cases of a light source and observer at finite distances.

2. *The Development in Spherical Harmonics.*

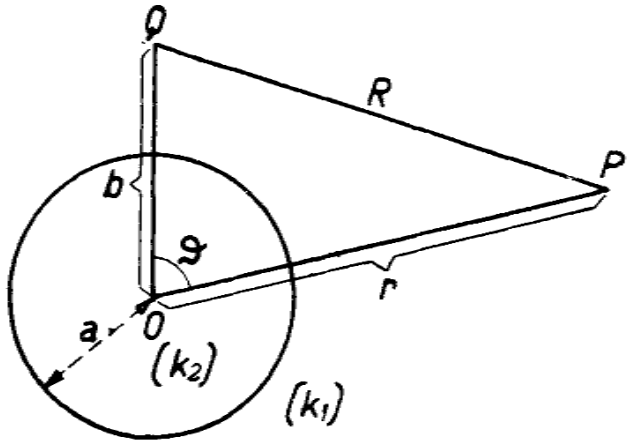
Using spherical coordinates, let (fig. 1) the electrical dipole be situated at $Q(b, o)$, the observer at $P(r, \vartheta)$, the radius of the sphere being a. The analytical problem consists of finding the function Π, representing the amplitude of the Hertzian vector divided by r, having a radial component only, and satisfying the following differential equations and boundary conditions (where $k_{1,2}^2$ has the usual meaning):

(1) $(\Delta+k_1^2)\Pi=0 \quad (r>a)$,
$(\Delta+k_2^2)\Pi=0 \quad (r<a)$,

(2) $\frac{\partial}{\partial r}(r\Pi)$ and $k^2\Pi$ continuous at $r=a$,

(3) a singularity at Q(b, o) of the form $\frac{e^{ik_1 R}}{ik_1 R}$ which is further called the "primary field." This means that in the outer space $\Pi=\Pi_{pr}+\Pi_{sec}$, where the "secondary

Fig. 1.



Coordinates used.

field" Π_{sec} has no singularities. The electric and magnetic fields are given by

$$E_r=\left(k^2+\frac{\partial^2}{\partial r^2}\right)(r\Pi)\,;\quad E_\vartheta=\frac{1}{r}\frac{\partial^2}{\partial r\partial\vartheta}(r\Pi)\,;$$

$$E_\phi=0,\ H_r=H_\vartheta=0\,;\quad H_\phi=-\frac{ic}{\omega}k^2\frac{\partial\Pi}{\partial\vartheta}\,;$$

$$k_{1,2}^2=\frac{\epsilon_{1,2}\omega^2+i\sigma_{1,2}\omega}{c^2}\,.$$

The known solution in the form of harmonic series is:

$$\Pi_{tot}=\Pi_{pr}+\Pi_{sec}=\frac{e^{ik_1R}}{ik_1R}$$
$$+\sum_{n=0}^{n=\infty}(2n+1)R_n\frac{\psi_n(k_1a)}{\zeta_n^{(1)}(k_1a)}\zeta_n^{(1)}(k_1b)\zeta_n^{(1)}(k_1r)P_n(\cos\vartheta)\ (r>a),$$

. . . (1a)

$$\Pi_{\text{tot}}=\frac{k_1^2}{k_2^2}\sum_{n=0}^{n=\infty}(2n+1)(1+\mathrm{R}_n)\frac{\psi_n(k_1a)}{\psi_n(k_2a)}\zeta_n^{(1)}(k_1b)\psi_n(k_2r)\mathrm{P}_n(\cos\vartheta)$$

$$(r<a), \quad . \quad . \quad (1b)$$

where the symbols have the following meaning :

$$\zeta_n^{(1)}(x)=\sqrt{\frac{\pi}{2x}}\,\mathrm{H}^{(1)}_{n+\frac{1}{2}}(x)=x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{e^{ix}}{ix}\right),$$

$$\zeta_n^{(2)}(x)=\sqrt{\frac{\pi}{2x}}\,\mathrm{H}^{(2)}_{n+\frac{1}{2}}(x)=x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{ie^{-ix}}{x}\right),$$

$$\psi_n(x)=\tfrac{1}{2}\{\zeta_n^{(1)}(x)+\zeta_n^{(2)}(x)\}$$

$$=\sqrt{\frac{\pi}{2x}}\,\mathrm{J}_{n+\frac{1}{2}}(x)=x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right),$$

(H=Hankel function), and where

$$\mathrm{R}_n=\frac{-\frac{1}{x}\left[\frac{d}{dx}\log\{x\psi_n(x)\}\right]_{x=k_1a}+\left[\frac{1}{x}\frac{d}{dx}\log\{x\psi_n(x)\}\right]_{x=k_2a}}{\left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(1)}(x)\}\right]_{x=k_1a}-\left[\frac{1}{x}\frac{d}{dx}\log\{x\psi_n(x)\}\right]_{x=k_2a}},$$

$$. \quad . \quad . \quad (2)$$

It may be remarked that (1 *a*) is symmetrical in *r* and *b*, expressing the well-known property of reciprocity, stating that transmitter and receiver may be interchanged.

3. *Transformation of* (1) *to a Sum of a Series of Residues and an Integral.*

For the derivation of (1) it is necessary to develop first the primary field in a harmonic series. It is well known that two different series are obtained according to whether $r>b$ or $r<b$, viz. :

$$\frac{e^{ik_1\mathrm{R}}}{ik_1\mathrm{R}}=\sum_{n=0}^{n=\infty}(2n+1)\zeta_n^{(1)}(k_1r)\psi_n(k_1b)\mathrm{P}_n(\cos\vartheta) \quad (r>b), \quad (3a)$$

$$\frac{e^{ik_1\mathrm{R}}}{ik_1\mathrm{R}}=\sum_{n=0}^{n=\infty}(2n+1)\zeta_n^{(1)}(k_1b)\psi_n(k_1r)\mathrm{P}_n(\cos\vartheta) \quad (r<b). \quad (3b)$$

If we express (1 *a*) completely as a harmonic series by substitution of (3 *b*) into (1 *a*) and considering the

total field at the surface of the sphere only ($r=a$), while making use of the Wronksian :

$$\psi_n(x)\zeta_n^{(1)\prime}(x)-\zeta_n^{(1)}(x)\psi_n'(x)=\frac{i}{x^2}, \quad . \; . \; . \quad (4)$$

we obtain

$$\Pi_{\text{tot}}=-\frac{i}{(k_1a)^3}\sum_{n=0}^{n=\infty}(2n+1)\frac{\zeta_n^{(1)}(k_1b)}{\zeta_n^{(1)}(k_1a)}\frac{1}{\mathrm{N}_n}\mathrm{P}_n(\cos\vartheta), \quad . \quad (5)$$

where

$$\mathrm{N}_n(x,y)=\left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(1)}(x)\}\right]_{x=k_1a} - \left[\frac{1}{y}\frac{d}{dy}\log\{y\psi_n(y)\}\right]_{y=k_2a}. \quad (6)$$

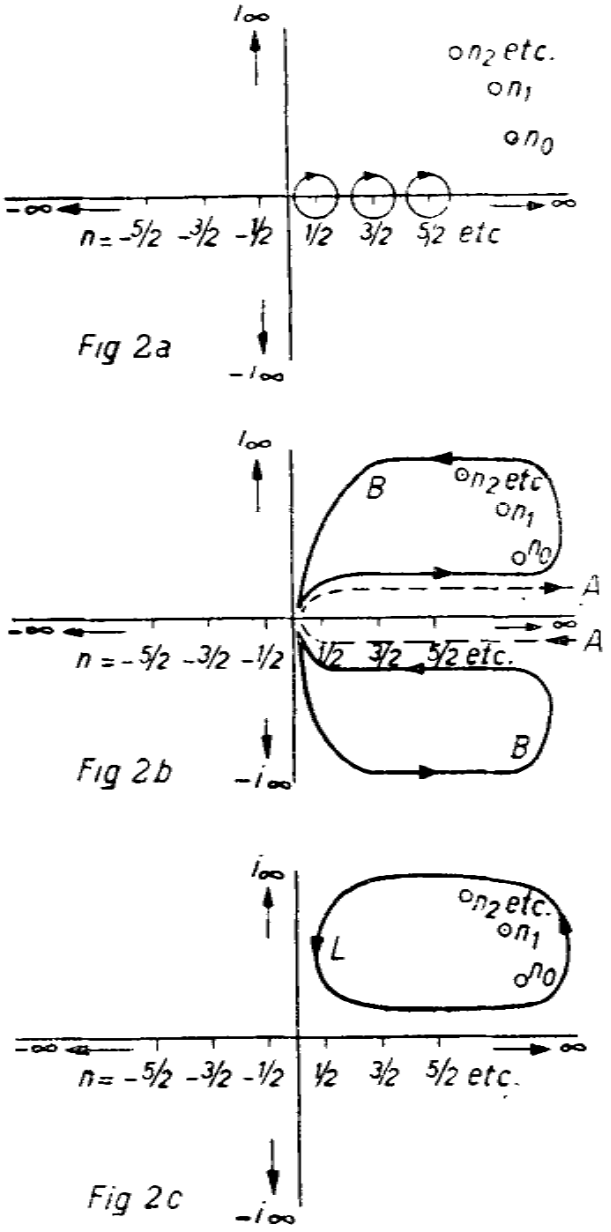
Following Watson, we transform (5) into a continuous integral over n leading to

$$\Pi_{\text{tot}}=\frac{1}{(k_1a)^3}\int_{\mathrm{L}}\frac{n\,dn}{\cos(n\pi)}\frac{\zeta_{n-1\,2}^{(1)}(k_1b)}{\zeta_{n-1\,2}^{(1)}(k_1a)}\frac{1}{\mathrm{N}_{n-1\,2}}\mathrm{P}_{n-1/2}\{\cos(\pi-\vartheta)\} + \frac{1}{(k_1a)^3}\int_0^{i\infty}\frac{n\,dn}{\cos(n\pi)}\frac{\zeta_{n-1\,2}^{(1)}(k_1b)}{\zeta_{n-1/2}^{(1)}(k_1a)}\left(\frac{1}{\mathrm{N}_{n-1/2}}-\frac{1}{\mathrm{N}_{-n-1/2}}\right) \times\mathrm{P}_{n-1/2}\{\cos(\pi-\vartheta)\}. \quad . \quad (7)$$

Here the contour of integration L in the complex n-plane lies wholly in the first quadrant and encloses all poles lying in that quadrant (see fig. 2 *c*). This path of integration actually makes the sum of the two integrals (7) identical with the series (5), which may be clear by the following transformation:—First the series (5) is transformed into an integral with the path of integration formed by small circles round the poles $n=1/2,\ 3/2,\ 5/2\ldots.$ of the function $1/\cos(n\pi)$ (see fig. 2 *a*, where $n_0, n_1, n_2\ldots.$ are the poles of the integrand apart from the factor $1/\cos(n\pi)$). Further, this path can be transformed to the loop A of fig. 2 *b*, extending from $\infty-i\epsilon$ to $\infty+i\epsilon$. As the integral becomes exponentially zero when $|n|$ tends to ∞, the integral with integration path A is the difference of the contour integral B of fig. 2 *b* and an integral along the imaginary axis. Further, it appears that the integrand has no poles in the minus first quadrant, so that B may be contracted to the contour L of fig. 2 *c*,

whereas the integral along the imaginary axis may be transformed to the second integral of (7) along the positive imaginary axis.

Fig. 2.



Transformation of the path of integration in the n-plane.

Apparently the second term of (7) vanishes when $N_{n-1/2}$ is an even function of n. Now the first term of (6) is even in n, whereas the second term, for the case of radiotelegraphy, is to a high approximation equal to

$$\left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_2a},$$

which is also an even function of n ; this approximation is a consequence of the fact that in practice $| e^{2ik_2a} |$ (corresponding to the attenuation of plane waves which have travelled over a distance equal to the diameter of the earth) is very small, due to the great absorption in the earth. Therefore, as a first approximation, we may consider $N_{n-1/2}$ as an even function of n and omit the second term of (7). Watson (*l. c.* p. 99) showed also for a numerical case that this term is negligible compared with the first term, and further on we shall neglect this term.

The first term of (7) enables us to obtain a very physical interpretation of a typical possibility inherent to our spherical problem and which has also been noticed by Watson. For if we develop $1/\cos(n\pi)$ as

$$\frac{1}{\cos(n\pi)} = 2e^{i\pi n} \sum_{m=0}^{m=\infty} e^{i\pi m(2n+1)}, \quad . \quad . \quad . \quad (8)$$

valid over the whole integration path L, we can write (7) as

$$\Pi_{\text{tot}} \sim \sum_{m=0}^{m=\infty} \Pi_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\Pi_m = \frac{2e^{i\pi m}}{(k_1a)^3} \int_L n\, dn\, e^{i\pi n(2m+1)} \frac{\zeta^{(1)}_{n-1/2}(k_1b)}{\zeta^{(1)}_{n-1/2}(k_1a)} \frac{1}{\overline{N}_{n-1/2}} \times P_{n-1/2}\{\cos(\pi - \vartheta)\}. \quad . \quad (10)$$

We are now able to represent this integral as the sum of the residues of all the enclosed poles being the zeros n_s of $N_{n-1/2}$ only, and therefore may write it as

$$\Pi_m = \frac{4\pi i e^{i\pi m}}{(k_1a)^3} \sum_{s=0}^{s=\infty} n_s e^{i\pi n_s(2m+1)} \frac{\zeta^{(1)}_{n_s-1/2}(k_1b)}{\zeta^{(1)}_{n_s-1/2}(k_1a)} \frac{1}{\left(\frac{\partial N_{n-1/2}}{\partial n}\right)_{n=n_s}} \times P_{n_s-1/2}\{\cos(\pi - \vartheta)\}. \quad . \quad (11)$$

Further, for all values of ϑ not too near 0 or π, and, as the modulus of all poles n_s is of the order k_1a (as will be shown later on), the well-known asymptotic formula for the spherical harmonics of complex order with positive imaginary part

$$P_{n-1/2}\{\cos(\pi - \vartheta)\} \sim \frac{1}{\sqrt{2\pi n \sin\vartheta}} e^{-in(\pi-\vartheta)+i\pi/4}, \quad (12)$$

may now be introduced as an approximation in the still rigorous expression (10), yielding

$$\Pi_m \sim \frac{2\sqrt{2\pi}\,.\,e^{i\pi(m+3/4)}}{(k_1 a)^3\sqrt{\sin\vartheta}} \times \sum_{s=0}^{s=\infty} \frac{\zeta^{(1)}_{n_s-1/2}(k_1 b)}{\zeta^{(1)}_{n_s-1/2}(k_1 a)} \frac{n_s^{1/2}}{\left(\frac{\partial N_{n-1/2}}{\partial n}\right)_{n=n_s}} e^{in_s(\vartheta+2\pi m)}. \quad . \quad (13)$$

The above mentioned physical interpretation is given directly by the last factor of (13), viz. :

$$e^{in_s(\vartheta+2\pi m)} = e^{(i\alpha_s-\beta_s)(\vartheta+2\pi m)} \quad . \quad . \quad . \quad (14)$$

putting

$$n_s = \alpha_s + i\beta_s \quad (\beta_s > 0).$$

Hence (13) represents a wave travelling round the sphere with a phase velocity and attenuation determined by α_s and β_s respectively. But the presence in the exponent of the combination $\vartheta+2\pi m$ clearly shows that the term Π_m of (9) is representative of a wave which has travelled from the source to the receiver only after having made m complete revolutions round the sphere. This explanation becomes especially vivid for the case of radiotelegraphy, where the transmitter is near the surface of the sphere. It will be clear that the contribution to the field at the surface of the earth by a wave which has travelled several times round the earth, is extremely small compared with the wave coming directly from the transmitter to the receiver ($m=0$). In Watson's analysis the first term Π_0 of the series (9) is retained only, and we will follow the same procedure in Part I. of this paper, so that we will further consider the approximation

$$\Pi_{tot} \sim \Pi_0 \sim \frac{2\sqrt{2\pi}e^{3\cdot 4i\pi}}{(k_1 a)^3\sqrt{\sin\vartheta}} \times \sum_{s=0}^{s=\infty} \frac{\zeta^{(1)}_{n_s-1/2}(k_1 b)}{\zeta^{(1)}_{n_s-1\,2}(k_1 a)} \frac{n_s^{1/2}}{\left(\frac{\partial N_{n-1\,2}}{\partial n}\right)_{n=n_s}} e^{in_s\vartheta}. \quad . \quad (15)$$

4. *Approximations for the Bessel Functions.*

It will be clear that for numerical interpretation of (15), a knowledge of the poles n_s and of the residues is necessary. We will therefore make use of *two* approximations :

(*a*) The Debye approximations, as given in full detail in the second edition of Jahnke-Emde, Funktionentafeln, pp. 204–207. These become in our case, where n/z and z lie in the first quadrant :

$$\zeta_n^{(1)}(z)=\sqrt{\frac{\pi}{2z}}\,\mathrm{H}^{(1)}_{n+1/2}(z)$$

$$\sim \frac{2}{i\sqrt{(n+\frac{1}{2})z}\sqrt{\frac{z^2}{(n+\frac{1}{2})^2}-1}}$$

$$\times\cos\left\{\frac{\pi}{4}+(n+\tfrac{1}{2})\int_1^{z/(n+1/2)}\sqrt{u^2-1}\,\frac{du}{u}\right\}, \quad . \quad (16a)$$

$$\zeta_n^{(2)}(z)=\sqrt{\frac{\pi}{2z}}\,\mathrm{H}^{(2)}_{n+1/2}(z)$$

$$\sim \frac{1}{\sqrt{(n+\frac{1}{2})z}\sqrt{\frac{z^2}{(n+\frac{1}{2})^2}-1}}$$

$$\times e^{i\pi/4-i(n+1/2)\int_1^{z/(n+1/2)}\sqrt{u^2-1}\,\frac{du}{u}}, \quad . \quad (16b)$$

where the root $\sqrt{\dfrac{z^2}{(n+\frac{1}{2})^2}-1}$ is defined as lying in the same quadrant as $z/(n+\frac{1}{2})$.

(*b*) The approximations reducing the Bessel functions of any order to these functions of order 1/3. For the orders of magnitude obtaining in our problem, these are

$$\mathrm{H}^{(1)}_{(2)n}(z)\sim\frac{\sqrt{\dfrac{z^2}{n^2}-1}}{\sqrt{3}}\,e^{\pm i\pi/6\mp in\int_1^{z/n}\frac{du}{u}(u^2-1)^{1/2}}$$

$$\times\mathrm{H}^{(1)}_{(2)1/3}\left\{\frac{n}{3}\left(\frac{z^2}{n^2}-1\right)^{3/2}\right\}, \quad . \quad (17)$$

where

$$-\pi/2<\arg\sqrt{\frac{z^2}{n^2}-1}<\pi/2.$$

The formulæ (16) are suitable everywhere except for z near n, while (17), valid only for $n>>1$, is of special use when z is near n. It is further easy to verify that the first term of the asymptotic development of $H_{1/3}$ occurring in (17) leads to (16).

Moreover (16 a) clearly shows the position of the zeros of $\zeta_n^{(1)}(z)$ at constant n, these being determined by the condition that the cosine becomes zero, similar roots not occurring in the quadrant considered in the case of $\zeta_n^{(2)}(z)$. Again, these zeros of $\zeta_n^{(1)}(z)$ separate the two regions where one or the other of the exponential parts of the cosine function dominates, these zeros requiring obviously

$$\mathrm{I}_n\left\{\frac{\pi}{4}+(n+\tfrac{1}{2})\int_1^{z/(n+1/2)}\sqrt{u^2-1}\,\frac{du}{u}\right\}=0, \quad . \quad (18)$$

where I denotes the imaginary part of its argument.

The close relationship between (16) and the well-known W.K.B. approximation is at once apparent, remembering that $z\zeta_n(z)$ satisfies the differential equation

$$\frac{d^2}{dz^2}(z\zeta_n)+\left\{1-\frac{n(n+1)}{z^2}\right\}.z\zeta_n=0. \quad . \quad . \quad (19)$$

5. *The Limiting Case of a Plane Boundary* ($a\to\infty$).

In this paragraph we will show how the well-known solution of the similar problem, but for a plane boundary, can be obtained as a limiting case of the spherical problem. We therefore start with (1 a) in the form

$$\Pi_{\mathrm{sec}}=\sum_{n=0}^{n=\infty}(2n+1)\mathrm{R}_n\frac{\psi_n(k_1a)}{\zeta_n^{(1)}(k_1a)}\zeta_n^{(1)}(k_1b)\zeta_n^{(1)}(k_1r)\mathrm{P}_n(\cos\vartheta)$$
$$(r>a), \quad . \quad (1a)$$

and have to consider the limit of this expression for $a\to\infty$ while keeping constant the three quantities

$b-a=z_q$ (height of source above boundary),

$z-a=z_p$ (height of receiver above boundary),

$a\vartheta=\rho$ (horizontal distance between source and receiver).

Using the method of § 3 in order to transform (1*a*) into an integral, one finds

$$\Pi_{\text{sec}} = -i\int_B \frac{n\,dn}{\cos(n\pi)} R_{n-1/2}\zeta^{(1)}_{n-1/2}(k_1a)\psi_{n-1/2}(k_1a) \times \frac{\zeta^{(1)}_{n-1/2}(k_1b)\zeta^{(1)}_{n-1/2}(k_1r)}{\{\zeta^{(1)}_{n-1/2}(k_1a)\}^2} P_{n-1/2}\{\cos(\pi-\vartheta)\}. \quad (20)$$

Now putting $n=a\lambda$, and taking the limit for $a\to\infty$, while keeping λ constant, one finds, for the different factors of the integrand of (19),

(*a*)

$$\operatorname*{Lim}_{a\to\infty} R_{n-1/2} = \frac{k_2^2\sqrt{\lambda^2-k_1^2}-k_1^2\sqrt{\lambda^2-k_2^2}}{k_2^2\sqrt{\lambda^2-k_1^2}+k_1^2\sqrt{\lambda^2-k_2^2}}. \quad (21)$$

Here we have used the limits of the logarithmic derivates of (17) in the form

$$\frac{d}{dz}\log\{z\zeta^{(1)}_n(z)\} \sim \frac{1}{2z} \mp i\frac{n}{z}\left(\frac{z^2}{n^2}-1\right)^{3/2} + e^{\pm 2/3 i\pi}\frac{z}{n}\sqrt{\frac{z^2}{n^2}-1}\,\frac{H^{(1)}_{2/3}\left\{\frac{n}{3}\left(\frac{z^2}{n^2}-1\right)^{3/2}\right\}}{H^{(1)}_{1/3}\left\{\frac{n}{3}\left(\frac{z^2}{n^2}-1\right)^{3/2}\right\}}, \quad (22)$$

where

$$-\frac{\pi}{2} < \arg\sqrt{\frac{z^2}{n^2}-1} < \frac{\pi}{2}.$$

In our limit z/n has the constant value k/λ, but the argument of $H_{1/3}$ and $H_{2/3}$ is tending to $+\infty$, and we obtain

$$\operatorname*{Lim}_{z=ka\to\infty}\frac{d}{dz}\log\{z\zeta^{(1)}_{n\lambda}(z)\} = \mp\frac{1}{k}\sqrt{\lambda^2-k^2},$$

where $\sqrt{\lambda^2-k^2}$ has always a positive real part. Further we have

$$\operatorname*{Lim}_{a\to\infty}\psi_n(ka) = \tfrac{1}{2}\operatorname*{Lim}_{a\to\infty}\zeta^{(2)}_n(ka),$$

because k lies in the first quadrant and therefore the $\zeta^{(1)}_n$ part vanishes, so that

$$\operatorname*{Lim}_{a\to\infty}\frac{d}{dz}\log\{z\psi_n(z)\} = \tfrac{1}{2}\operatorname*{Lim}_{a\to\infty}\frac{d}{dz}\log\{z\zeta^{(2)}_n(z)\}.$$

The limit of $R_{n-1/2}$ then follows directly from the definition (2) of R_n.

(*b*)

$$\operatorname*{Lim}_{a\to\infty} \zeta^{(1)}_{n-1/2}(k_1 a)\psi_{n-1/2}(k_1 a) = \frac{1}{2ik_1 a^2\sqrt{\lambda^2-k_1^2}},$$

because, as in (*a*), we first have

$$\operatorname*{Lim}_{a\to\infty} \zeta^{(1)}_{n-1/2}(k_1 a)\psi_{n-1\,2}(k_1 a) = \tfrac{1}{2}\operatorname*{Lim}_{a\to\infty} \zeta^{(1)}_{n-1/2}(k_1 a)\zeta^{(2)}_{n-1/2}(k_1 a),$$

and then can apply (16), using in (16 *a*) only the dominant term of the cos-function.

(*c*)

$$\operatorname*{Lim}_{\substack{a\to\infty \\ b-a=z_q}} \frac{\zeta^{(1)}_n(k_1 b)}{\zeta^{(1)}_n(k_1 a)} = e^{-\sqrt{\lambda^2-k_1^2}z_q}.$$

This limit follows directly from the dominant term in (16 *a*), because

$$\frac{\zeta^{(1)}_{n-1/2}(k_1 b)}{\zeta^{(1)}_{n-1/2}(k_1 a)} \sim \frac{\sqrt{a}}{\sqrt{b}}\,\frac{\sqrt[4]{k_1^2a^2-n^2}}{\sqrt[4]{k_1^2b^2-n^2}}\,e^{ia\lambda\int_{k_1/\lambda}^{\{1+z_q/a\}k_1/\lambda}\sqrt{u^2-1}\,\frac{du}{u}},$$

and this expression leads at once for $a\to\infty$ to the above limit.

(*d*) Similarly we have

$$\operatorname*{Lim}_{\substack{a\to\infty \\ b-a=z_p}} \frac{\zeta^{(1)}_n(k_1 r)}{\zeta^{(1)}_n(k_1 a)} = e^{-\sqrt{\lambda^2-k_1^2}z_p}.$$

(*e*)

$$\operatorname*{Lim}_{\substack{a\to\infty \\ I(n)>0}} \frac{P_{n-1/2}\{\cos(\pi-\vartheta)\}}{\cos(n\pi)} - \operatorname*{Lim}_{\substack{a\to\infty \\ I(n)<0}} \frac{P_{n-1/2}\{\cos(\pi-\vartheta)\}}{\cos(n\pi)} = 2iJ_0(\lambda\rho),$$

where $I(n)$ means the imaginary part of n.

We may prove this by using the relation

$$\frac{P_{n-1/2}\{\cos(\pi-\vartheta)\}}{\cos(n\pi)} = tg(n\pi)P_{n-1/2}(\cos\vartheta) + \frac{2}{\pi}Q_{n-1/2}(\cos\vartheta),$$

which is valid for unrestricted n but real ϑ. For $n=a\lambda$ tending to ∞ the functions $P_{n-1/2}(\cos\vartheta)$ and $Q_{n-1/2}(\cos\vartheta)$ have each the same limit for $I(n)>0$ and $I(n)<0$, but for $tg\,(n\pi)$ we have

$$\operatorname*{Lim}_{|n|\to\infty} tg(n\pi) = \pm i \text{ resp. for } I(n) \gtrless 0.$$

Therefore

$$\lim_{\substack{a\to\infty \\ \mathrm{I}(n)>0}} \frac{\mathrm{P}_{n-1/2}\{\cos(\pi-\vartheta)\}}{\cos(n\pi)} - \lim_{\substack{a\to\infty \\ \mathrm{I}(n)<0}} \frac{\mathrm{P}_{n-1/2}\{\cos(\pi-\vartheta)\}}{\cos(n\pi)}$$

$$= 2i \lim_{n\to\infty} \mathrm{P}_{n-1/2}(\cos\vartheta)$$

$$= 2i \lim_{n\to\infty} \mathrm{P}_{n-1/2}\left(\cos\frac{\lambda\rho}{n}\right) = 2i\mathrm{J}_0(\lambda\rho).$$

After substitution of the limits a—d in (20), which are valid both for the part of B above and the part below the real axis, we can reduce the path of integration to the real axis after applying (e), and finally obtain

$$\Pi_{\mathrm{sec}} = \frac{1}{ik_1}\int_0^\infty \frac{\lambda d\lambda}{\sqrt{\lambda^2-k_1^2}} \frac{(k_2^2\sqrt{\lambda^2-k_1^2} - k_1^2\sqrt{\lambda^2-k_2^2})}{(k_2^2\sqrt{\lambda^2-k_1^2} + k_1^2\sqrt{\lambda^2-k_2^2})} \times e^{-\sqrt{\lambda^2-k_1^2}(z_q+z_p)}.\mathrm{J}_0(\lambda\rho).$$

This coincides with the expression given as (1) in a former paper [10], where the primary field was defined as $e^{ik_1\mathrm{R}}/\mathrm{R}$ instead of $e^{ik_1\mathrm{R}}/ik_1\mathrm{R}$ as here. Hence a complete agreement is obtained, including a finite height of both point source and receiver and a finite conductivity of the two media.

6. *Development in Characteristic Functions.*

It may be noted that the solution used so far is quite distinct from the well-known method of development in Characteristic Functions, as, *e. g.*, is a standard method for string problems and in wave mechanics.

In our problem we have two differential equations for $r>a$ and $r<a$; before applying the theory of Characteristic Functions, we shall first consider the more general equation of the Sturm-Liouville type

$$\{\Delta + g(x, y, z)\}\Pi' = f(x, y, z).$$

This equation has Characteristic Functions defined by the following properties :

1. $(\Delta+\lambda_j^2 g)\phi_j = 0$,
2. ϕ_j finite for $r=0$ and $r=\infty$.

The expansion of the solution in Characteristic Functions as described for instance in chapter v., § 2 of Courant und

Hilbert, *Methoden der Mathematischen Physik*, i., here becomes

$$\Pi'(\mathrm{P})=\Sigma_j \frac{\int \phi_j . f . d\tau}{(1-\lambda_j^2)} \phi_j(\mathrm{P}). \quad . \quad . \quad . \quad . \quad (23)$$

Now let $g(x, y, z)$ degenerate to the discontinuous function $g=k_1^2$ for $r>a$ and $g=k_2^2$ for $r<a$, $g\Pi$ and $\partial/\partial r\ (r\Pi)$ remaining continuous for $r=a$; then the new Characteristic Functions are determined by the conditions

$$\left.\begin{array}{ll} (1) & (\Delta+k_1^2\lambda_j^2)\phi_j=0 \quad (r>a) \\ & (\Delta+k_2^2\lambda_j^2)\phi_j=0 \quad (r<a), \\ (2) & k^2\phi_j \text{ and } \dfrac{\partial}{\partial r}(r\phi_j) \text{ continuous for } r=a, \\ (3) & k_1^2\displaystyle\int_{r=a}^{r=\infty} \phi_j^2 d\tau + k_2^2 \int_{r=0}^{r=a} \phi_j^2 d\tau = 1. \end{array}\right\} \quad (24)$$

We now proceed by a method which has been fully expounded by A. Sommerfeld [11] in a classical paper, where the primary field is $\cos(k_1\mathrm{R})/(ik_1\mathrm{R})$, belonging to a point source of stationary waves with a singularity at Q; this field leads to a three-dimensional impulsive function $4\pi i\delta(\mathrm{R})/k_1$ at Q for $f(x, y, z)$. Hence substitution in (23) yields

$$\Pi'_{\mathrm{tot}}=-\frac{4\pi}{ik_1}\Sigma_j \frac{\phi_j(\mathrm{Q})\phi_j(\mathrm{P})}{(1-\lambda_j^2)}. \quad . \quad . \quad . \quad (25)$$

As the Characteristic Functions should everywhere be finite and single-valued, they are of the form

$$\phi_{n,j}=\left\{\begin{array}{ll} \mathrm{A}_{n,j}\zeta_n^{(1)}(\lambda_{n,j}k_1 r)\mathrm{P}_n(\cos\vartheta) & (r>a), \\ \mathrm{B}_{n,j}\psi_n(\lambda_{n,j}k_2 r)\mathrm{P}_n(\cos\vartheta) & (r<a), \end{array}\right\}. \quad (26)$$

where n is a positive integer.

The boundary conditions further require

$$k_1^2\mathrm{A}_{n,j}\zeta_n^{(1)}(\lambda_{n,j}k_1 a)=k_2^2\mathrm{B}_{n,j}\psi_n(\lambda_{n,j}k_2 a), \quad . \quad (27\,a)$$

$$\mathrm{A}_{n,j}\left[\frac{d}{dz}\{z\zeta_n^{(1)}(z)\}\right]_{z=\lambda_{n,j}k_1 a}=\mathrm{B}_{n,j}\left[\frac{d}{dz}\{z\psi_n(z)\}\right]_{z=\lambda_{n,j}k_2 a}, \quad . \quad . \quad . \quad (27\,b)$$

so that

$$\left|\begin{array}{ll} k_1^2a^2\zeta_n^{(1)}(\lambda_{n,j}k_1 a) & k_2^2a^2\psi_n(\lambda_{n,j}k_2 a) \\ \left[\dfrac{d}{dz}\{z\zeta_n^{(1)}(z)\}\right]_{z=\lambda_{n,j}k_1 a} & \left[\dfrac{d}{dz}\{z\psi_n(z)\}\right]_{z=\lambda_{n,j}k_2 a} \end{array}\right|=0,$$

$$\text{or } \mathrm{N}_n(\lambda_{n,j}k_1 a,\ \lambda_{n,j}k_2 a)=0, \quad (28)$$

(see (6)).

The difference between the method used so far and the present method of Characteristic Functions now becomes very clear by the fact that, although both methods consider the zeros of N,—which is a function of n and the ka's—the first method considers the zeros as a function of n for constant ka's, whereas the method of Characteristic Functions treats the zeros as a function of the ka's, while keeping the integer n constant.

Introducing the normalization condition (3) of (24), and remembering that the integration over all space has to be performed for the inner space and outer space separately, we obtain

$$\frac{A_{n,j}^2}{k_1}\int_{\lambda_{n,j}k_1a}^{\infty}\{\zeta_n^{(1)}(u)\}^2u^2\,du + \frac{B_{n,j}^2}{k_2}\int_0^{\lambda_{n,j}k_2a}\{\psi_n(u)\}^2u^2\,du=\frac{(2n+1)}{4\pi}\lambda_{n,j}^3. \quad (29)$$

(27), (28), and (29) now determine completely $A_{n,j}$ and $B_{n,j}$; after reduction, and making use of well-known integral properties of the Bessel functions, we get

$$A_{n,j}^2=\frac{(2n+1)}{2\pi k_1^2a^3\{\zeta_n^{(1)}(x)\}^2\left[\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}\right)\log\{x\zeta_n^{(1)}(x)\} - \frac{k_1^2}{k_2^2}\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}\right)\log\{y\psi_n(y)\}\right]} \quad \ldots \quad (30a)$$

$$B_{n,j}^2=\frac{(2n+1)}{2\pi k_2^2a^3\{\psi_n(y)\}^2\left[\frac{k_2^2}{k_1^2}\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}\right)\log\{x\zeta_n^{(1)}(x)\} - \left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}\right)\log\{y\psi_n(y)\}\right]} \quad \ldots \quad (30b)$$

where $x=\lambda_{n,j}k_1a$ and $y=\lambda_{n,j}k_2a$,
so that after substitution in (25) we obtain

$$\left.\begin{aligned}\Pi'_{\text{tot}}&=\frac{4\pi i}{k_1}\sum_{n=0}^{n=\infty}\Sigma_j\frac{A_{n,j}^2}{(1-\lambda_{n,j}^2)}\zeta_n^{(1)}(\lambda_{n,j}k_1b)\zeta_n^{(1)}(\lambda_{n,j}k_1r)P_n(\cos\vartheta)\\ &\qquad (r>a),\\ \Pi'_{\text{tot}}&=\frac{4\pi i}{k_1}\sum_{n=0}^{n=\infty}\Sigma_j\frac{A_{n,j}B_{n,j}}{(1-\lambda_{n,j}^2)}\zeta_n^{(1)}(\lambda_{n,j}k_1b)\psi_n(\lambda_{nj}k_2r)P_n(\cos\vartheta)\\ &\qquad (r<a);\end{aligned}\right\} \quad \ldots \quad (31)$$

(31) gives the solution of our problem as a double series of Characteristic Functions, the summations to be extended over the integer values of n and over all the j values every time belonging to a given n. It will be clear that (31) must be equal to (1). Further, the orthogonality of the functions $P_n(\cos\vartheta)$ requires that for a fixed n the summation over the corresponding j values must every time lead to the coefficients of $P_n(\cos\vartheta)$ in the harmonic series (1). This idea will further be developed for the special case $|k_2|=\infty$ in paragraph 7 *c*.

7 *a*. *The Harmonic Series for a Perfectly Reflecting Sphere.* ($|k_2|=\infty$).

In this paragraph we will investigate the limiting case for a perfectly reflecting sphere and therefore $|k_2|=\infty$, while basing ourselves on the harmonic series. In the next § 7 *b* we elaborate the residue series and obtain numerical results, showing also the influence of the finite height $b-a=h$ of the transmitter above the surface of the sphere. In § 7 *c* we compare the residue series with the development in Characteristic Functions.

The harmonic series (5) for $a=b=r$ and $|k|_2=\infty$ becomes

$$\Pi_{\mathrm{tot}}=\frac{i}{z^2}\sum_{n=0}^{n=\infty}(2n+1)\frac{1}{z\mathrm{N}_n(z)}\mathrm{P}_n(\cos\vartheta)_{(z=k_1a)}, \quad . \quad (32)$$

where

$$\frac{1}{z\mathrm{N}_n(z)}=\left[\frac{z\zeta_n^{(1)}(z)}{\frac{d}{dz}\{z\zeta_n^{(1)}(z)\}}\right]_{z=k_1a}, \quad . \quad . \quad (33)$$

and we will now examine the numerical structure of the coefficient $1/z\mathrm{N}(z)$ as a function of the positive integer n for $|z|>>1$. To this end we can at once make use of (22), where, as $|z|>>1$, the first term may be neglected. We therefore have

$$z\mathrm{N}_n(z)\sim -i\frac{n}{z}\left(\frac{z^2}{n^2}-1\right)^{3/2} + e^{2/3i\pi}\frac{z}{n}\sqrt{\frac{z^2}{n^2}-1}\,\frac{\mathrm{H}_{2/3}^{(1)}\left\{\frac{n}{3}\left(\frac{z^2}{n^2}-1\right)^{3/2}\right\}}{\mathrm{H}_{1/3}^{(1)}\left\{\frac{n}{3}\left(\frac{z^2}{n^2}-1\right)^{3/2}\right\}}, \quad (34)$$

where

$$-\frac{\pi}{2}<\arg\sqrt{\frac{z^2}{n^2}-1}<\frac{\pi}{2}.$$

The square root remains perfectly well defined even for z^2/n^2 real, when this variable is considered as having an infinitesimal positive imaginary part. The approximate expression (34) can, when z is not too near n, be further considerably simplified, for then the argument of the Hankel functions remains so great that the first term of their asymptotic development suffices. Under these circumstances (34) degenerates to the simple form

$$\frac{1}{z\mathrm{N}_n(z)}\approx\frac{1}{i\sqrt{1-\frac{n^2}{z^2}}}, \quad . \quad . \quad . \quad . \quad (35)$$

with the square root in the first quadrant, and this form coincides with the first W.K.B. approximation of the proper solution of (19) Although (34) was developed for $n>>1$, (35) still gives the right value for $n=0$, viz.

$$\frac{1}{z\mathrm{N}_0(z)}=-i.$$

Further (35) degenerates for $n>>>z$ to

$$\frac{1}{z\mathrm{N}_n(z)}\approx-\frac{z}{n}.$$

Moreover, for the transition case ($n\sim z$) the second term of (34) dominates by far, and when $n=z$ exactly; in the latter case this second term becomes indefinite, but, while taking the first term of the power series of the Hankel functions, it is easy to show that

$$\left\{\frac{1}{z\mathrm{N}_n(z)}\right\}_{n=z}=\frac{2}{6^{1/3}}\frac{\Pi(1/3)}{\Pi(2/3)}e^{-2/3i\pi}z^{1/3}$$

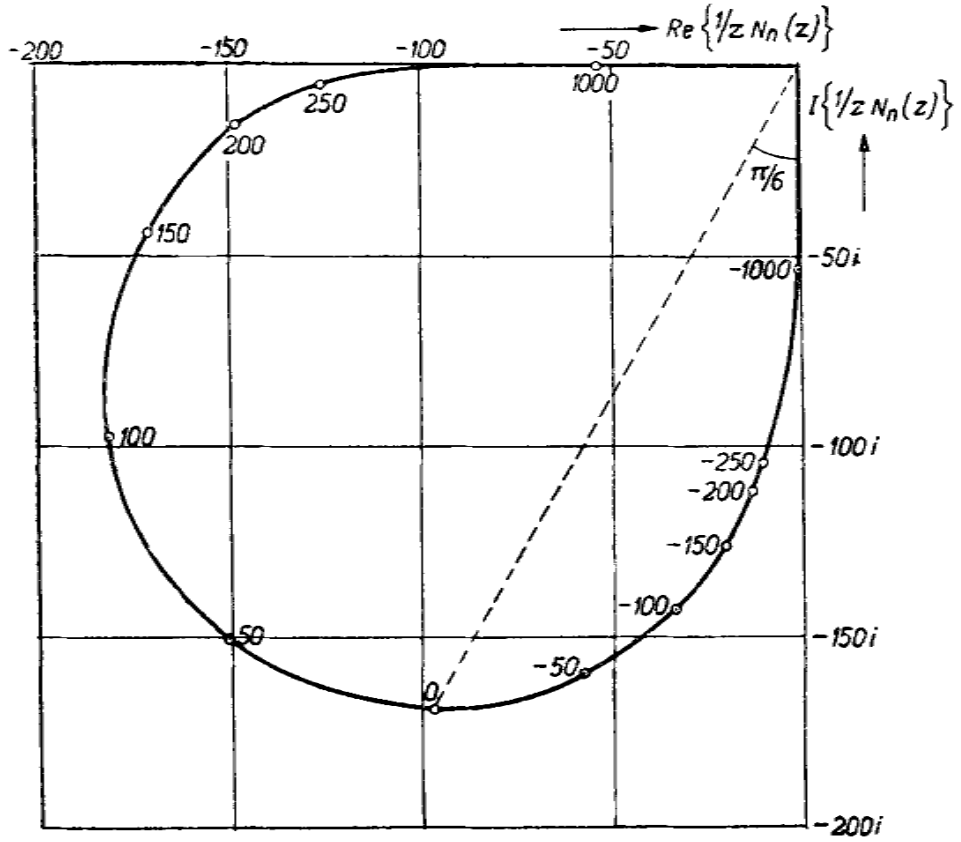
$$=1{\cdot}0888\ldots e^{-2/3i\pi}z^{1/3}.$$

The above approximations enable us to determine numerically $1/z\mathrm{N}_n(z)$ as a function of n in the complex plane. We have chosen $z=k_1a=5{\cdot}72.10^6$ corresponding to waves of 7 meter in the radio case ($2\pi a=4.10^9$ cm.). In fig. 3 we give the result numerically correct. The numbers in this figure refer to the values of $n-z$. It appears from this figure that (a) the modulus of $1/z\mathrm{N}_n(z)$

has a maximum near $n=z$; (*b*) the density along the curve is rather small where the modulus of $1/z\ \mathrm{N}_n(z)$ is considerable, whereas nearly all points are lying on the two extreme parts of the curve near the two axes; (*c*) the curve has a horizontal tangent at $n=z$.

Again, figs. 4 and 5 give the modulus and the argument of $1/z\mathrm{N}_n(z)$ as a function of $n-z$; a.o. this clearly demonstrates (*b*) above.

Fig. 3.



Coefficient of the harmonic series for $|k| = \infty$ and $z=k_1a=5{\cdot}72.10^6$.

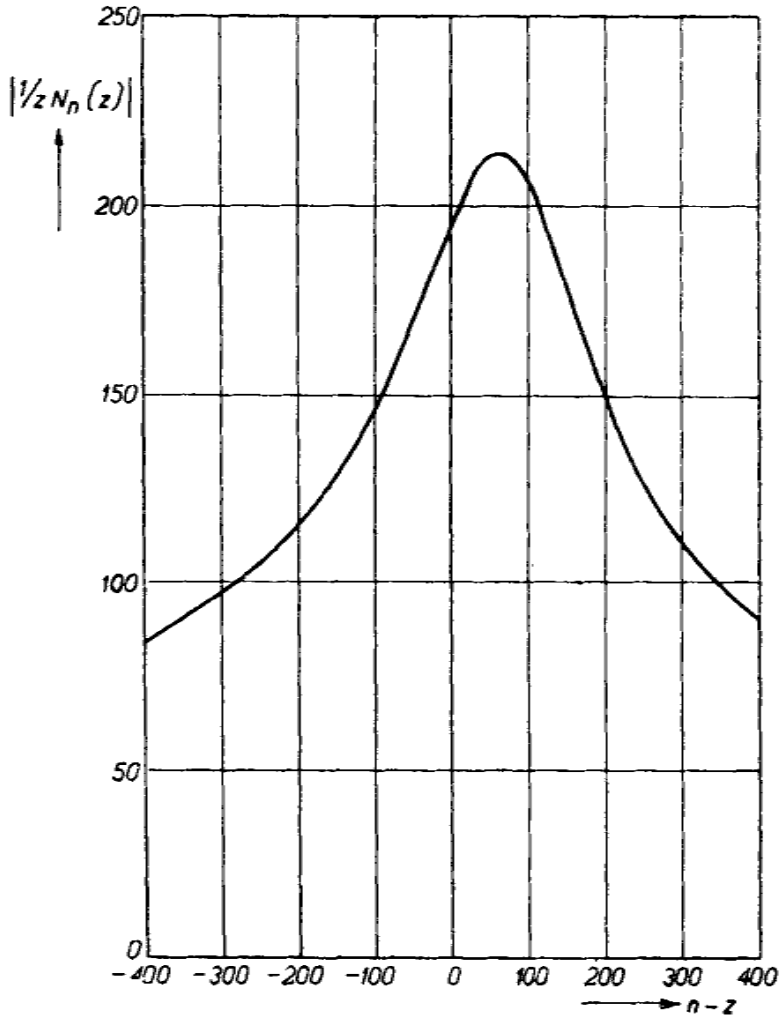
A further numerical consideration shows that for all values of n where $1/z\mathrm{N}_n(z)$ has an appreciable value, *i. e.*, where $n\sim z$, a good approximation to this function is given by the second term of (34) only; moreover, in this range z^2/n^2-1 may be replaced by $2(z/n-1)$, so that the final approximation for (34) gives $1/z\mathrm{N}_n(z)$ as a function of the only variable $\sigma=(n-z)/z^{1/3}$:

$$\frac{1}{z\mathrm{N}_n(z)} \approx z^{1/3} e^{-2/3 i\pi} \frac{\mathrm{H}^{(1)}_{1/3}\{\tfrac{1}{3}(-2\sigma)^{3/2}\}}{\sqrt{-2\sigma}\,\mathrm{H}^{(1)}_{2/3}\{\tfrac{1}{3}(-2\sigma)^{3/2}\}}. \quad . \quad (36)$$

This shows the universal character of the figures 3, 4, and 5, which retain the same form also for other values of the wave-length than $\lambda = 7$ meter.

Returning to the harmonic series (32), the coefficients of which have now been completely investigated, it follows that, when ϑ is not near zero, the oscillatory character of $P_n(\cos\vartheta)$ for fixed ϑ as a function of n causes the series

Fig. 4.



Modulus of the coefficient of fig. 3 as a function of n.

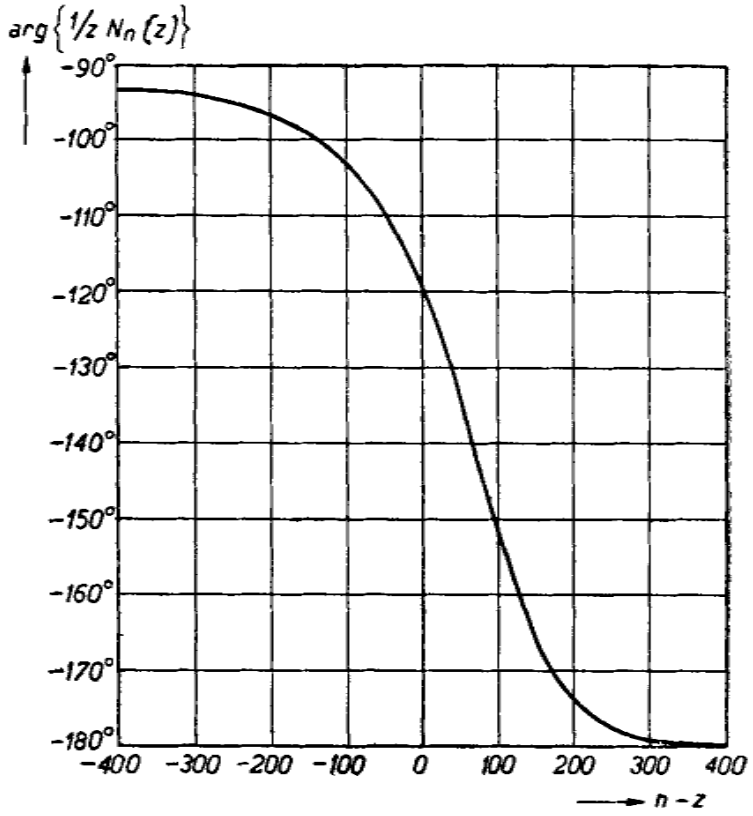
to oscillate very rapidly, so that successive terms nearly cancel each other, which makes the summation of (32) a very intricate process. We will now turn our attention to the much more rapidly convergent residue-series.

7b. *The Series of Residues for* $|k_2| = \infty$.

Consider (7). For $|k_2| = \infty$ the expression $N_{n-1/2}$ is, according to (33), even in n, so that the second term of

(7) is exactly zero. We can therefore further use the series of residues as given by (15) and for numerical applications a knowledge of the zeros of $N_{n-1/2}$, as well as the values of $\frac{\partial N_{n-1/2}}{\partial n}$ at these zeros is necessary. A further investigation shows that for $|z| >> 1$, as is the case in the radio application, the zeros n_s of $N_{n-1/2}$ as a function of n for constant z have a modulus at least of

Fig. 5.



Argument of the coefficient of fig. 3 as a function of n.

the order $|z|$. In order therefore to find the proper approximation for $N_{n-1/2}(z)$ we differentiate (16 a) logarithmically, remembering that both $|z| >> 1$ and $|n| >> 1$. We thus find

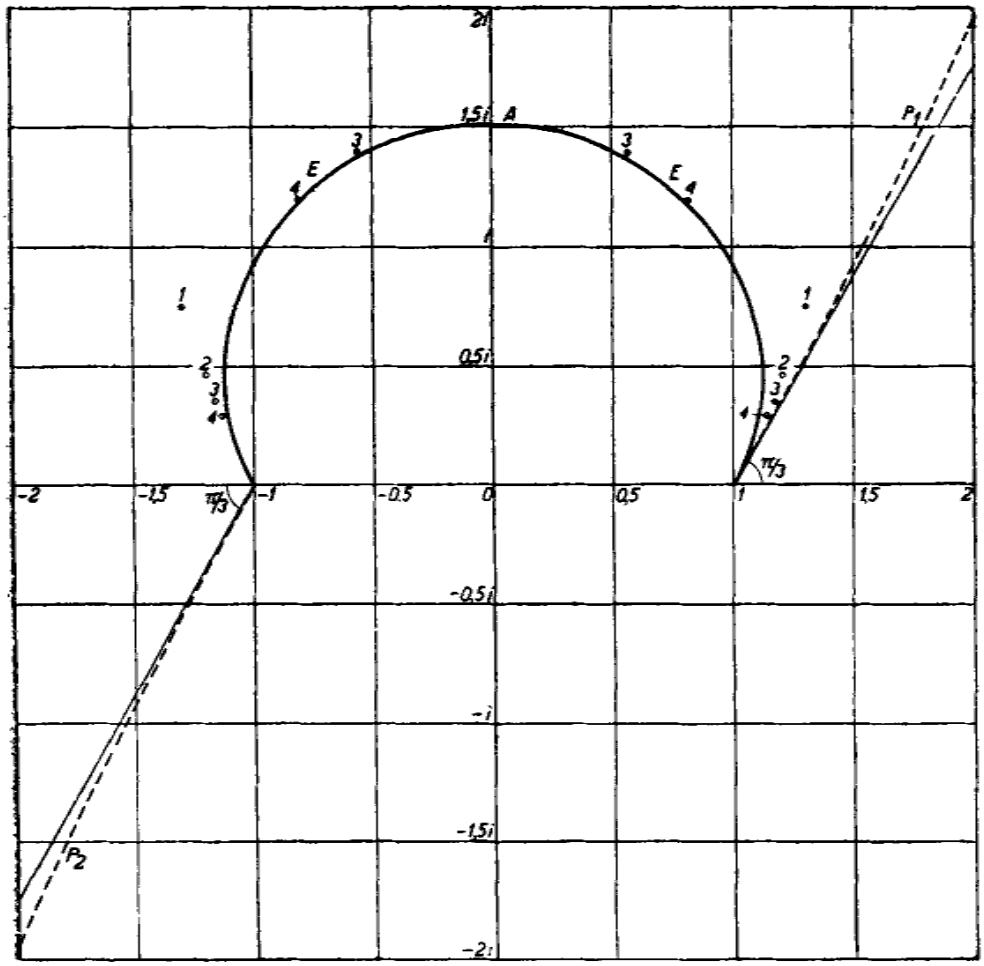
$$N_n(z) \sim -\frac{n}{z^2}\sqrt{\frac{z^2}{n^2}-1}\,.\,tg\left\{\frac{\pi}{4}+(n+\tfrac{1}{2})\int_1^{z/n}\sqrt{u^2-1}\,\frac{du}{u}\right\}, \quad . \quad . \quad . \quad (37)$$

so that the zeros n_s of $N_{n-1/2}(z)$ are given by the transcendental equations

$$i(n_s+\tfrac{1}{2})\int_1^{z/ns}\sqrt{1-u^2}\,\frac{du}{u}=\pi(s+\tfrac{1}{4}) \quad (s=0,\ 1,\ 2,\ \ldots). \qquad \ldots \quad (38)$$

For real $z=k_1a$ (meaning a non-absorbing atmosphere)

Fig. 6.



Poles and characteristic frequencies in the n/z plane for $|k_2|=\infty$.

it follows from (38) that with the approximations used all the zeros n_s are situated on the curves

$$\mathrm{Re}\left\{\frac{n}{z}\int_1^{z/n}\sqrt{1-u^2}\,\frac{du}{u}\right\}=0 \qquad \ldots \quad (39)$$

in the n/z plane, which are shown dotted in fig. 6 as P_1 and P_2. The poles n_s nearest the real axis and on the branch P_1, with the exception of the first few, are easily

found from (38), remembering that $|z/n-1|$ is a small quantity. They are given by

$$n_s=z+\tau_s z^{1/3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where

$$\tau_s=\tfrac{1}{2}\{3\pi(s+\tfrac{1}{4})\}^{2/3}e^{i\pi/3}. \quad . \quad . \quad . \quad . \quad (40\,a)$$

In order to calculate the first few roots, which lie in the immediate neighbourhood of $n=z$, other authors, *e. g.*, Nicholson [12], Laporte [7], have used approximations with more terms. The difficulty is that especially the first root, which is of predominant influence on the numerical results, lies in the boundary region of the two Debye approximations for the Hankel functions; see, *e. g.*, Jahnke-Emde, 'Funktionentafeln' (1933), pp. 204, 207. In this way they found, in our notation,

$$\text{Nicholson} \ldots \quad \tau_0=0{\cdot}804e^{i\pi/3}.$$

$$\text{Laporte} \ldots . \quad \tau_0=0{\cdot}81\ e^{i\pi/3}.$$

However, it is of interest to note that the expression (36), which is valid near $n=z$, yields as equation for the first few zero

$$H^{(1)}_{2/3}\{\tfrac{1}{3}(-2\sigma)^{3/2}\}=H^{(1)}_{2/3}\left\{\frac{2^{3/2}}{3\sqrt{z}}(n-z)^{3/2}e^{-3/2i\pi}\right\}=0.$$

Remembering further that

$$H^{(1)}_{2/3}(\alpha e^{-i\pi})=\frac{e^{-5/6i\pi}}{\sin\pi/3}\{J_{2/3}(\alpha)-J_{-2/3}(\alpha)\}, \quad . \quad (41$$

and calling α_s the zeros of (41), which are all positive, and identifying

$$\frac{2^{3/2}}{3\sqrt{z}}(n_s-z)^{3/2}e^{-3/2i\pi}=\alpha_s e^{-i\pi},$$

we again find for the zeros n_s the relation (40), but, instead of (40 *a*),

$$\tau_s=\tfrac{1}{2}(3\alpha_s^{\frac{1}{2}})^{2/3}e^{i\pi/3}, \quad . \quad . \quad . \quad . \quad . \quad (40\,b)$$

which furnishes a better approximation than (40 *a*).

Summarizing, the equations (40 *a*) and (40 *b*) thus yield for the first three roots

from (40 *a*):	*from* (40 *b*):
$\tau_0=0{\cdot}885e^{\pi/5}$,	$\tau_0=0{\cdot}808\,e^{i\pi/3}$,
$\tau_1=2{\cdot}589\,e^{i\pi/3}$,	$\tau_1=2{\cdot}577\,e^{i\pi/3}$,
$\tau_2=3{\cdot}830\,e^{i\pi/3}$,	$\tau_2=3{\cdot}824\,e^{i\pi/3}$,

so that in a relatively simple way we find from (40 *b*) a value for τ_0 which is very near the accurate values already given by Nicholson and Laporte.

Having thus found the position of the first few poles n_s, the factor of the corresponding residues,

$$\left(\frac{\partial \mathrm{N}}{\partial n}\right)_{n=n_s} = -2\tau_s z^{-1/3},$$

may be obtained from (37) and (40). Substitution of the results in (15) yields the final and rapidly converging expression

$$\Pi_{\substack{r=a \\ |k_2|=\infty}} \sim 2\frac{e^{iz\vartheta}}{iz\vartheta}\sqrt{\frac{\pi}{2}}iz^{1/3}\vartheta \cdot \sum_s \frac{\zeta^{(1)}_{z+z^{1/3}\tau_s}(k_1 b)}{\zeta^{(1)}_{z+z^{1/3}\tau_s}(k_1 a)}\frac{e^{iz^{1/3}\tau_s\vartheta}}{\tau_s}. \quad (42)$$

Taking the special case of the transmitter being placed on the surface of the sphere ($b=a$), (42) simplifies to

$$\Pi_{\substack{r=a=b \\ |k_2|=\infty}} \sim 2\frac{e^{iz\vartheta}}{iz\vartheta}\mathrm{F}(\eta), \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\mathrm{F}(\eta) = \sqrt[3]{\frac{\pi^2}{2}}e^{i\pi/4}\sqrt{\eta}\sum_s \frac{e^{i(2\pi)^{1/3}\tau_s\eta}}{\tau_s}, \quad . \quad . \quad (43\,a)$$

$$\eta = \sqrt[3]{\frac{k_1 a}{2\pi}}\vartheta \sim \frac{\mathrm{D}}{a^{2/3}\lambda^{1/3}},$$

D being the distance between transmitter and receiver measured along the surface of the sphere.

The structure of (43), which can be written

$$\Pi_{\substack{r=a=b \\ |k_2|=\infty}} \sim 2\frac{e^{ik_1\mathrm{D}}}{ik_1\mathrm{D}}\mathrm{F}(\eta), \quad . \quad . \quad . \quad . \quad (44)$$

clearly shows the correction factor $\mathrm{F}(\eta)$ (being a function of *one* variable only and which is due to the curvature), which must be added to the solution $2e^{ik_1\mathrm{D}}/ik_1\mathrm{D}$, which itself belongs to the plane problem ; the factor 2 represents the total reflexion by the infinitely conducting second medium ; (44) confirms the result first obtained by Watson (*l. c.*).

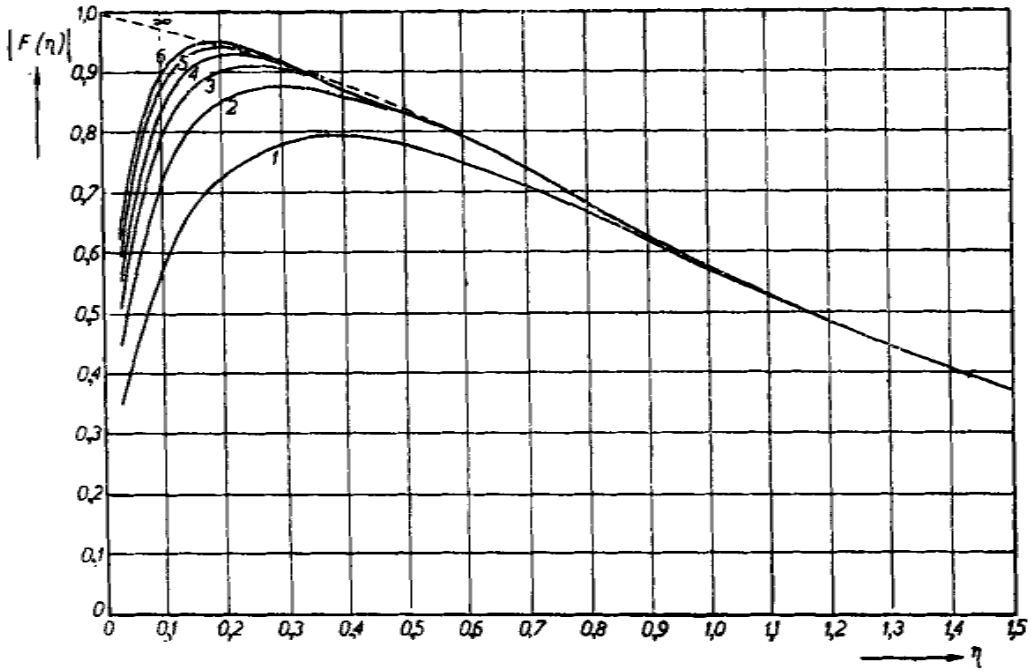
It is of interest to note that the variable η introduced above is closely related with the phase difference ϕ of two beams, one going straight to the receiver through the sphere, the other going along the surface. In fact for

small ϑ this phase difference is given by $\phi = \frac{\eta^3}{24}$, and thus the solution (44) in this respect shows a great analogy with other diffraction problems where also the field near the boundary of the optical shadow is expressible as a function of the corresponding phase difference.

From the expressions (43 a) and (44) for the vector Π the normal component of the electric field can simply be calculated with the relations given in § 2.

The result (43 a) is plotted in fig. 7, where the curves

Fig. 7.



Modulus of the sum of the first few terms of F (η).

marked 1, 2, 3 . . . 6 represent approximations of the modulus of F(η) as a function of η calculated with 1 2, 3 . . . 6 terms respectively, and where due consideration has been given to the phases of the different terms of the series.

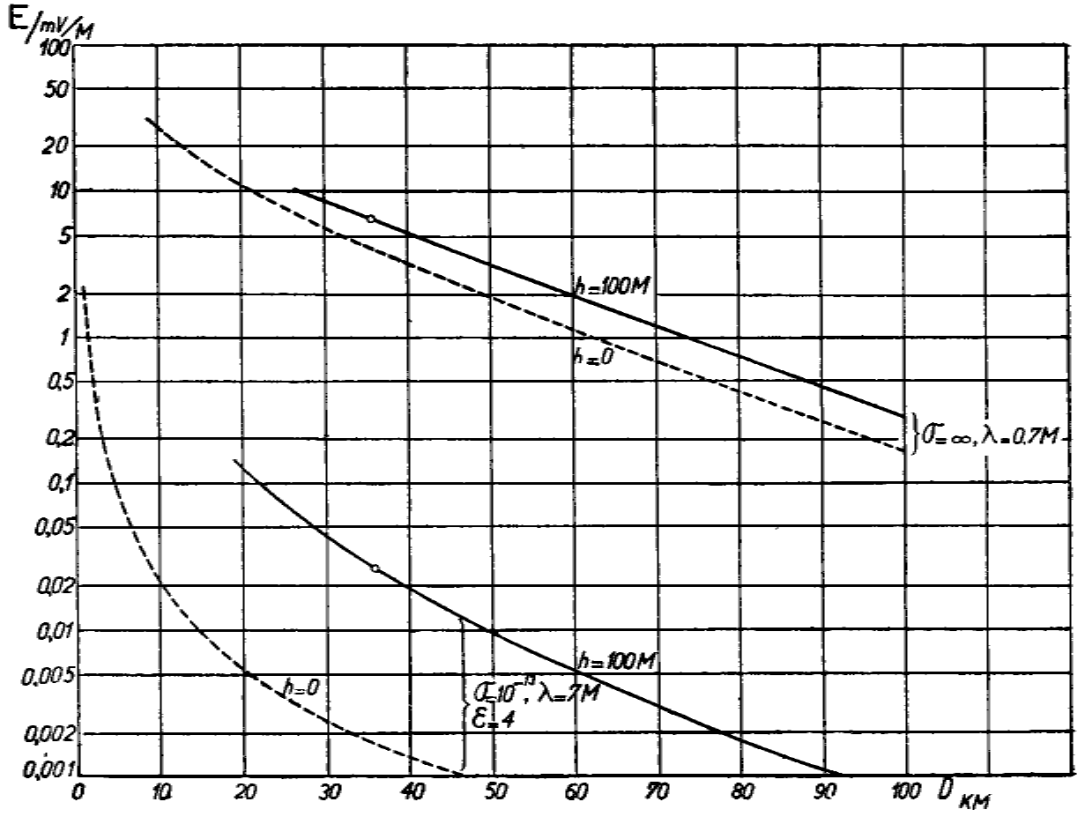
Moreover, referring to (42), where the transmitter was not yet lowered to the surface of the earth, this formula has been used to calculate the curve labelled $h = 100$ m., $\sigma = \infty$, $\lambda = 0{\cdot}7$ metre in fig. 8. As indicated in this figure it was constructed for a wave-length of 70 cm. and for a height of the transmitter above the earth of $h = b - a =$ 100 metre. As a comparison we give in the same figure the curve marked $h = 0$ for the same wave-length and which was calculated from (44).

In calculating these curves, as for all following ones, the moment of the transmitter has been assumed to be such that, when placed on an infinitely conducting plane earth, would radiate 1 kw.

7 c. *The Characteristic Functions and Values for* $|k_2| = \infty$.

The results of § 6 become much simpler for the special case $|k_2| = \infty$. The coefficients $B_{n,j}$, and therefore

Fig. 8.



Influence of the height of the transmitter field on the
for 1 kw. " radiated."

also the Characteristic Functions $\phi_{n,j}$, then vanish all for $r<a$. Moreover (28), defining the Characteristic Values $\lambda_{n,j}k_1$, then simplifies to

$$\left[\frac{d}{dz}\{z\zeta_n^{(1)}(z)\}\right]_{z=z_{n,j}=\lambda_{n,j}k_1 a} = 0, \quad . \quad . \quad . \quad (45)$$

whose roots, according to the definition of $\zeta_n^{(1)}(z)$, correspond to the zeros of the polynomials

$$e^{-iz}\frac{d}{dz}\left\{ z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{e^{iz}}{iz}\right)\right\} = 0 ; \quad . \quad (46)$$

hence, remembering that $z\zeta_n^{(1)}(z)$ satisfies (19), the expression (30 a) can be reduced to

$$A_{n,j}^2 = \frac{(2n+1)z_{n,j}^2}{2\pi k_1^2 a^3 \{\zeta_n^{(1)}(z_{n,j})\}^2 \{n(n+1)-z_{n,j}^2\}},$$

and the series (31) for $b=a$ to

$$\Pi'_{\text{tot}} = \frac{i}{2k_1 a} \sum_{n=0}^{n=\infty} (2n+1) P_n(\cos\vartheta) \sum_{j=0}^{j=n} \frac{z_{n,j}}{\{n(n+1)-z_{n,j}^2\}} \times \left\{ \frac{1}{k_1 a - z_{n,j}} - \frac{1}{k_1 a + z_{n,j}} \right\}. \quad (47)$$

This series corresponds to a point source $\cos(k_1 R)/(ik_1 R)$, representing stationary waves, while the series (32) corresponds to a point source $e^{ik_1 R}/(ik_1 R)$, representing diverging waves only. In order to transform (32) to a double series analogous to (47) we first write it as

$$\Pi_{\text{tot}} = \frac{i}{k_1 a} \sum_{n=0}^{n=\infty} (2n+1) P_n(\cos\vartheta) \cdot \left[\frac{\zeta_n^{(1)}(z)}{\frac{d}{dz}\{z\zeta_n^{(1)}(z)\}} \right]_{z=k_1 a} \cdot \quad . \quad . \quad . \quad (48)$$

Now it may be remarked that the definition of $\zeta_n^{(1)}z)$, as given in § 2, implies that $z^{n+1}e^{-iz}\zeta_n^{(1)}(z)$ is a polynomial of degree n, and that therefore the last factor of (48) can be written as a quotient of two polynomials of degree n and $n+1$ respectively. As the denominator is of higher degree than the numerator, this quotient can be split up in partial fractions, which, using the differential equation (19), can be given the form contained in the following series :

$$\Pi_{\text{tot}} = \frac{i}{k_1 a} \sum_{n=0}^{n=\infty} (2n+1) P_n(\cos\vartheta) \times \sum_{j=0}^{j=n} \frac{z_{n,j}}{\{n(n+1)-z_{n,j}^2\}} \frac{1}{(k_1 a - z_{n,j})}, \quad (49)$$

which consists of twice the first part of (47), corresponding to the diverging part of the stationary wave represented by this latter equation.

In order to consider (48) in more detail a discussion of the roots $z_{n,j}$ is necessary, to which we will now proceed.

Like in § 7 *b* we therefore have again to consider the roots of (37), but this time considered as a function of z, n being a positive integer. Therefore we now obtain for the equation determining the roots $z_{n,j}$ instead of (38)

$$i(n+\tfrac{1}{2})\int_1^{z_{n,j}/n}\sqrt{1-u^2}\,\frac{du}{u}=\pi(j+\tfrac{1}{4}),\quad (j=0,1,\ldots,n),\qquad (50)$$

where, as indicated, j goes from 0 to n, because in this way we cover all the $n+1$ roots of (46). For the same reason the summation over j in (47) and (49) extends also from 0 to n. It will be evident that all the roots of (50) are situated on the curve in the complex n/z plane given by

$$\mathrm{Re}\left\{\int_1^{z/n}\sqrt{1-u^2}\,\frac{du}{u}\right\}=0,\quad .\ .\ .\qquad (51)$$

instead of (39).

This curve is represented in fig. 6 by the drawn symmetrical curve E, on which therefore all the roots $z_{n,j}$ are situated approximately, whereas, as shown in § 7 *b*, the dotted curves P_1 and P_2 contained all the roots n_s of the equation (38). It is of considerable interest to observe that the curves E and P_1 have in $n/z=1$ a common point and a common tangent making an angle of $\pi/3$ with the real axis, as can be easily shown by comparing (39) and (51).

Further away from the point $n/z=1$ the curve E remains finite, whereas P_1 goes to infinity. In fact, by developing (39) for $|n/z|>>1$ it can easily be shown that the curve P_1 is asymptotically given by

$$\rho=\tfrac{1}{2}e^{1+\phi\,\mathrm{tg}\,\phi},$$

where ρ and ϕ are polar coordinates in the n/z plane.

A further consideration of (51) shows that the curve E cuts the imaginary axis in a point A (fig. 6) situated at $1{\cdot}509\ldots i=\alpha i$, say, where α is the real root of

$$\mathrm{cotgh}\left(\sqrt{1+\frac{1}{\alpha^2}}\right)=\sqrt{1+\frac{1}{\alpha^2}}$$

(see, *e. g.*, Jahnke-Emde, 'Funktionentafeln,' p. 31). It can further be shown that for n even there exist roots $z_{n,j}=z_{n,n/2}$ lying exactly on the imaginary axis corresponding to Characteristic Functions containing a purely aperiodic time function. The n/z values of these

"aperiodic" roots converge rapidly towards the value given above, as could be verified numerically by determining these roots exactly from (46) for small values of n; for these small values, however, it is better to consider $(n+\frac{1}{2})/z$ instead of n/z, according to (16 a). In fact, we obtained the following values:

$$2\tfrac{1}{2}/z_{2,1}=1{\cdot}566\,i,$$
$$4\tfrac{1}{2}/z_{4,2}=1{\cdot}526\,i,$$
$$6\tfrac{1}{2}/z_{6,3}=1{\cdot}517\,i,$$
$$8\tfrac{1}{2}/z_{8,4}=1{\cdot}514\,i,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\lim_{n\to\infty} n/z_{n,n/2}=1{\cdot}509\,i.$$

We also calculated the other roots for $n=1$, 2, 3, and 4 [13], and found

$$\begin{cases} z_{1,0}= 0{\cdot}866-0{\cdot}500\,i, \\ z_{1,1}=-0{\cdot}866-0{\cdot}500\,i. \end{cases} \qquad \begin{cases} z_{2,0}= 1{\cdot}808-0{\cdot}702\,i, \\ z_{2,1}= 0-1{\cdot}596\,i, \\ z_{2,2}=-1{\cdot}808-0{\cdot}702\,i. \end{cases}$$

$$\begin{cases} z_{3,0}= 2{\cdot}758-0{\cdot}843\,i, \\ z_{3,1}= 0{\cdot}871-2{\cdot}157\,i, \\ z_{3,2}=-0{\cdot}871-2{\cdot}157\,i, \\ z_{3,3}=-2{\cdot}758-0{\cdot}843\,i. \end{cases} \qquad \begin{cases} z_{4,0}= 3{\cdot}715-0{\cdot}954\,i, \\ z_{4,1}= 1{\cdot}752-2{\cdot}571\,i, \\ z_{4,2}= 0-2{\cdot}949\,i, \\ z_{4,3}=-1{\cdot}752-2{\cdot}571\,i, \\ z_{4,4}=-3{\cdot}715-0{\cdot}954\,i. \end{cases}$$

The $(n+\frac{1}{2})/z$ values of all these roots have also been marked with their n value in fig. 6. It is seen that even for these lowest values of n the roots lie already very near the universal E, which was derived for $n>>1$.

Returning to (50), using a similar procedure as in § 7 b, we obtain as a general expression for the roots near the real axis instead of (40)

$$z_{n,j}=n-\tau_j n^{1/3}, \qquad \ldots\ldots \quad (52)$$

where, as in (40 a),

$$\tau_j=\tfrac{1}{2}\{3\pi(j+\tfrac{1}{4})\}^{2/3}e^{i\pi/3}. \qquad \ldots \quad (52\,a)$$

Having thus obtained the values of the most important roots $z_{n,j}$, we are now in a position to apply this result to transform (49). We therefore invert in (49) the order

of summation of n and j and substitute the values of the roots $z_{n,j}$ as given by (52). Retaining the first order terms in n, (49) thus becomes

$$\Pi_{\text{tot}} \sim -\frac{i}{2k_1 a} \sum_j \frac{1}{\tau_j} \sum_{n=0}^{n=\infty} \frac{(2n+1)\mathrm{P}_n(\cos\vartheta)}{n^{1/3}\{n-k_1 a-(k_1 a)^{1/3}\tau_j\}}. \quad (53)$$

As before, we can transform (53) to an integral over n with a path of integration L as in fig. 2 *c*, where now the only enclosed pole is

$$n_0 = k_1 a + (k_1 a)^{1/3}\tau_j,$$

so that, taking the residue of this single pole, (53) becomes

$$\Pi_{\text{tot}} \sim -\frac{\pi i}{k_1 a} \sum_j \frac{{n_0}^{2/3}\mathrm{P}_{n_0-1/2}\{\cos(\pi-\vartheta)\}}{\tau_j \cos(n_0\pi)},$$

and, with the approximation (12) for P_n, we are led again to the series (42), but for $b=a$.

This result means that the s'th term of (42) is the sum of the contributions of the Characteristic Functions $\phi_{n,s}$, where for fixed s the summation extends over all values of n from 0 to ∞ —*e. g.*, the first and dominating term of (42) ($s=0$) is obtained by adding all Characteristic Functions $\phi_{n,0}$ which are just those which for fixed n have the smallest damping.

8. *Strongly Absorbing Sphere*

$$\left\{ \left| \frac{{k_2}^2}{(k_1 a)^{1/3} k_1 \sqrt{{k_2}^2 - {k_1}^2}} \right| << 1 \right\}.$$

In order to obtain, as in § 7 *b*, the series of residues, but now for finite k_2, a knowledge of the most important poles, *i. e.*, those in the n/z plane nearest the real axis, is again necessary. In these poles again $\mathrm{N}_{n-1/2}=0$, and according to the definition (6) and the approximation (37) we have for $|n/z|$ at least of the order unity

$$\mathrm{N}_{n-1/2}(z, z') \sim -\frac{n}{z^2}\sqrt{\frac{z^2}{n^2}-1}\,.\,tg\left\{\frac{\pi}{4}+(n+\tfrac{1}{2})\int_1^{z/n}\sqrt{u^2-1}\,\frac{du}{u}\right\} - \frac{1}{z'}\frac{d}{dz'}\log\{z'\psi_n(z')\}, \quad . \quad (54)$$

where

$$z = k_1 a \text{ and } z' = k_2 a.$$

From (54) one can at once deduce the relation which must be satisfied by the zeros n_s:

$$i(n_s+\tfrac{1}{2})\int_1^{z/n_s}\sqrt{1-u^2}\frac{du}{u}=\pi(s+\tfrac{3}{4})$$

$$-\text{arc } tg\left[\frac{z}{z'}\frac{\sqrt{1-\frac{n_s^2}{z^2}}}{\frac{d}{dz'}\log\{z'\psi_n(z')\}}\right], \qquad (55)$$

$$(s=0,\ 1,\ 2,\ \ldots)$$

where

$$0<\text{Re}\{\text{arc } tg[\quad]\}<\pi.$$

Now an approximation, derived in the same way as (35), and especially suited when $z'=k_2a$ has a great imaginary part, becomes

$$\frac{d}{dz'}\log\{z'\psi_n(z')\}\sim\frac{d}{dz'}\log\{z'\cdot\tfrac{1}{2}\zeta_n^{(2)}(z')\}$$

$$\approx\frac{d}{dz'}\log\{z'\zeta_z^{(2)}(z')\}\approx-i\sqrt{1-\frac{k_1^2}{k_2^2}}, \qquad (56)$$

as the zeros n_s of $N_{n-1/2}$ appear to lie near $z=n$. For strong absorption $|k_2|$ and $|k_1|$ are of the same order of magnitude, and for the zeros near $n/z=1$ the argument of arc tg in (55) is very small, and hence the function arc tg itself, so that as a first approximation this term may be neglected in (55). Thus (55) becomes nearly equal to (38), *i. e.*, the zeros near $n/z=1$ are again on the curve E of fig. 6; but a comparison of (55) and (38) shows that the zeros are now interspaced between the zeros belonging to the case $|k_2|=\infty$. This fact is especially clear when one considers the zeros in the immediate neighbourhood of $n/z=1$, and which are of primary numerical importance. Similarly to (40) and (40 *a*) these zeros are given by

$$n_s=z+\tau_s'z^{1/3}, \qquad (57)$$

with

$$\tau_s'=\tfrac{1}{2}\{3\pi(s+\tfrac{3}{4})\}^{2/3}e^{i\pi/3}. \qquad (57\,a)$$

As in § 7 *b*, we again retain in (15) the residues at these poles. In order further to find $\frac{\partial N}{\partial n}$ for $n=n_s$

we make use of (54), and derive from it, remembering (57) :

$$N\sim-\frac{\sqrt{-2\tau'}}{z^{1/3}}\,.\,tg\left\{\frac{\pi}{4}+\frac{(-2\tau')^{3/2}}{3}\right\}-\frac{1}{z'}\frac{d}{dz'}\log\{z'\psi_z(z')\}.$$

From this expression we derive $\frac{\partial N}{\partial n}$ for $n=n_s$, making use of the fact that $N_{n_s-1/2}=0$; thus one obtains three terms, of which one dominates for $|z|>>1$, so that we are left with

$$\frac{\partial N}{\partial n}\sim\frac{z}{z'^2}\left[\frac{d}{dz'}\log\{z'\psi_z(z')\}\right]^2.$$

Substitution in (15) gives, after reduction, whereby (57) is used, for our final expression :

$$\Pi_{abs}\sim 2\frac{e^{iz\vartheta}}{iz\vartheta}\cdot\frac{ik_2^4}{k_1^3(k_2^2-k_1^2)D}\,.\,2\pi e^{-i\pi/4}\eta^{3/2}\sum_s\frac{\zeta^{(1)}_{z+\tau_s'z^{1/3}}(k_1b)}{\zeta^{(1)}_{z+\tau_s'z^{1/3}}(k_1a)}e^{i(2\pi)^{1/3}\tau_s'\eta}\,. \qquad (58)$$

$$(z=k_1a)$$

This series shows a great analogy with (42), as $\eta=D/(a^{2/3}\lambda^{1/3})$ has the same meaning as before.

If we first limit ourselves to a consideration of the case $b=a$, we observe that (58) consists of *three* factors :

(1) the first factor

$$2\frac{e^{iz\vartheta}}{iz\vartheta}=2\frac{e^{ik_1D}}{ik_1D},$$

corresponding to the plane problem and for $|k_2|=\infty$;

(2) the second factor

$$\frac{ik_2^4}{k_1^3(k_2^2-k_1^2)D},$$

representing the effect of the absorption already found before for the plane problem ; for, with the primary field as defined in this paper, the plane problem was in a former paper [14] already reduced to the exact result

$$\Pi_{abs,\,a=\infty}=\frac{2k_2^3i}{(k_2^4-k_1^4)}\int_{k_1/h'}^{k_2/h'}\frac{e^{ih'Du}}{D}\,.\,d\left(\frac{1}{\sqrt{u^2-1}}\right),$$

$$\left(\frac{1}{h'^2}=\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)$$

and a first approximation for great k_1D is obtained from this expression by integrating it once by parts and

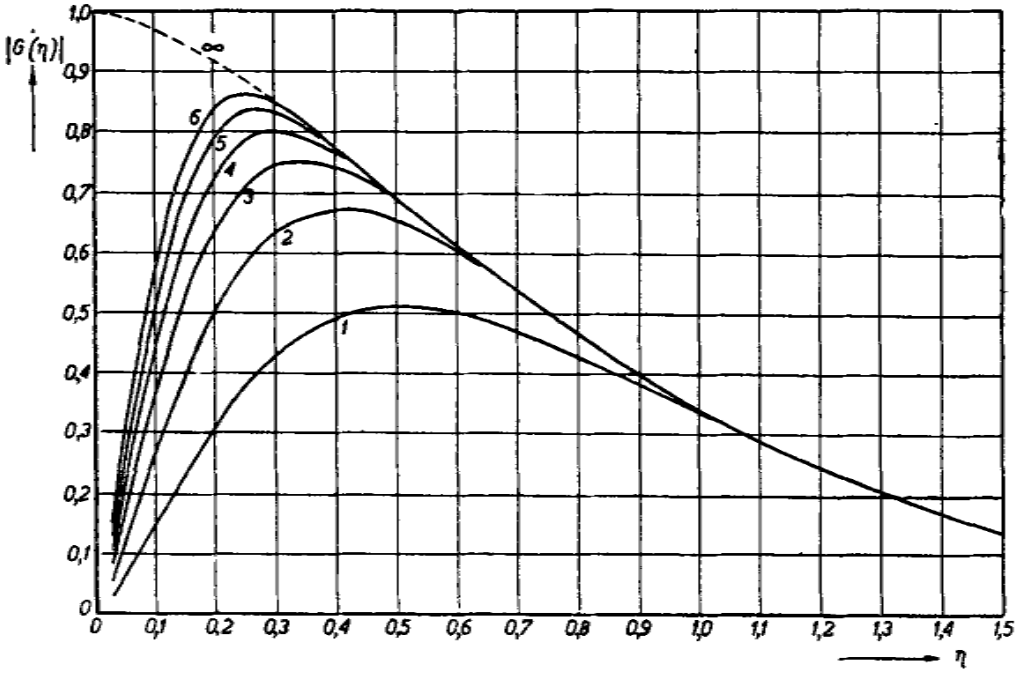
neglecting the contribution due to the upper limit of integration. We thus obtain exactly the product of the first two factors of (58).

(3) The third factor of (58) we call $G(\eta)$, and, according to the preceding, this factor represents the effect of the curvature of the earth.

Hence, when the transmitter is placed on the surface of the sphere, *i. e.*, for $b=a$, we can write (58) in the elegant form

$$\frac{\Pi_{\text{abs, } a\neq\infty}}{\Pi_{\text{abs, } a=\infty}} \sim G(\eta) = 2\pi e^{-i\pi/4}\eta^{3/2}\sum_s e^{i(2\pi)^{1/3}\tau_s'\eta}. \quad . \quad (59)$$

Fig. 9.



Modulus of the sum of the first few terms of G (η).

Thus the functions $F(\eta)$ (of § 7 *b*) and $G(\eta)$ (as given by (59)) give the correction factors due to the curvature of the earth, $F(\eta)$ being valid for $\sigma_2=\infty$ and $G(\eta)$ for strong absorption.

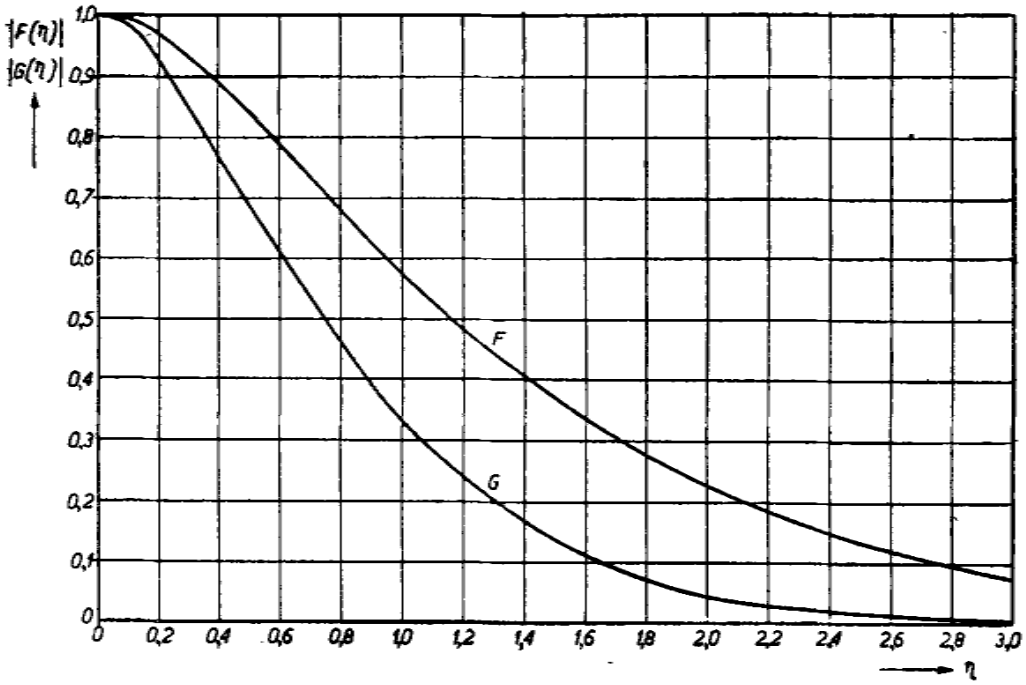
In fig. 9 we have represented by the curves 1, 2, 3 . . . 6 the approximations one finds for $|G(\eta)|$ if 1, 2 . . . 6 terms in (59) are retained; and thus fig. 9 for $G(\eta)$ is analogous to fig. 7 for $F(\eta)$. As our final result we give in fig. 10 the values of the two correction factors $|F(\eta)|$ and $|G(\eta)|$ applicable to the case $\sigma_2=\infty$ and great absorption respectively.

In general, and especially for short waves, not too great values of η are of primary importance: *e. g.*, for

$\lambda=7$ m., D$=50$ km., we have $\eta=0{\cdot}76$. Hence practical values of η are of the order unity, and in this range of values of η the difference of $|F(\eta)|$ and $|G(\eta)|$ is not very great. In practical cases the actual correction factor will be found between these two limiting curves; but, *e. g.*, for $\lambda=7$ m. and sea-water the actual values are already very near the curve $|G(\eta)|$. This results from the fact that then the argument of the arc *tg* function (55) is already very small, as the factor

$$\frac{k_2^2}{(k_1 a)^{1/3} k_1 \sqrt{k_2^2 - k_1^2}}$$

Fig. 10.



Correction factors for the curvature of the earth.

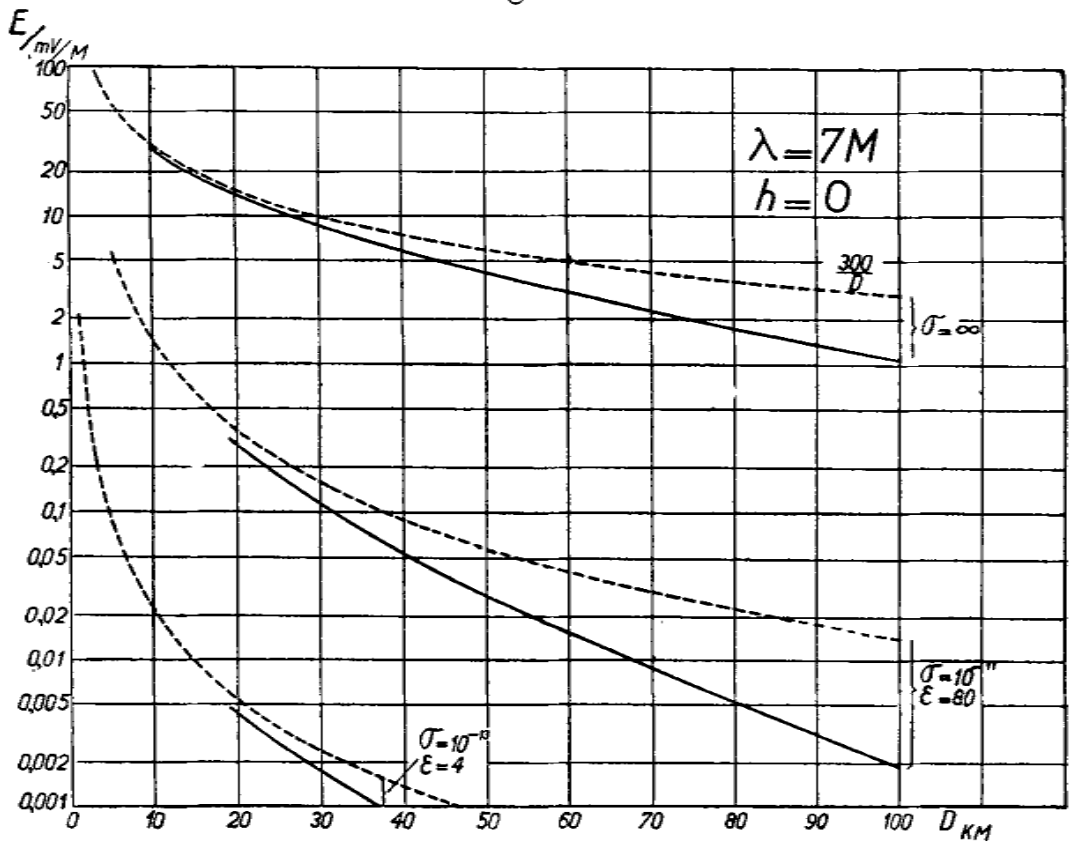
has a small modulus, and the first term of the arc *tg* development in (55) suffices for the approximation (59).

In this way the curves of fig. 11 (for $\lambda=7$ m.) were calculated. In this figure the drawn curves represent the electric field of an emitter radiating 1 kw. (for definition see § 7 *b*) for $\sigma_2=\infty$, for $\sigma_2=10^{-11}$ and $\epsilon_2=80$ (sea-water), and for $\sigma_2=10^{-13}$, $\epsilon_2=4$ (dry soil) respectively. The dotted lines represent the corresponding fields one would obtain for the same electrical constants of the earth, but for a plane surface. This figure shows very clearly that for $\lambda=7$ m. the influence of the finite conductivity of the soil dominates by far the influence of

the curvature of the earth for the distances considered. This is a general result for short waves.

Finally, it was possible with the aid of (58) to calculate the field of an emitter placed at a height $h=b-a$ above the ground. This was done for $\lambda=7$ m., $h=100$ m., and $\sigma_2=10^{-13}$, $\epsilon_2=4$. The result is given by the two lower curves of fig. 8, where the upper curves gave the result for $\lambda=0{\cdot}7$ m. and $\sigma_2=\infty$. In both cases no apparent effect is seen to occur at the optical horizon of the transmitter, this being situated at D=35·7 km. A further

Fig. 11.



Influence of the conductivity of the soil on the field for 1 kw. " radiated."

investigation showed, however, that for much shorter waves (of the order of 1 mm.) a distinct shadow effect makes its appearance (*cf.* Part II.).

Within a short time we hope to publish Part II. of this paper.

Summary.

The problem of diffraction of electromagnetic waves radiating from a point source outside an absorbing sphere

is treated by several methods. In Part I. an exposition of the method of Watson is given. This method consists in a transformation of the infinite series of spherical harmonics for the Hertzian vector into another series, which is much more rapidly converging for wave-lengths small compared with the dimensions of the sphere. Extensive numerical results are given for the case that the point source is a transmitter of radio waves, which may be situated on the surface of the in general finitely conducting sphere as well as at a small height above it. The former method is compared with that of development in characteristic functions, which is analogous to methods used in string problems and wave mechanics.

In Part II. the original harmonic series for the Hertzian vector is split up into an infinite sum of other harmonic series. Each of them may be considered as the effect of waves (degenerating into rays for extremely high frequencies) which are once or more times reflected at the inner surface of the sphere. These series may be approximated by Kelvin's principle of stationary phase as extended by Debye and Brillouin; this method gives appropriate approximations for the field of a radio transmitter in the region above its optical horizon, whereas, when the point source is at infinity it yields and extends Airy's well-known results for the intensity of the light in a rainbow.

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